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Remarks on a RG approach to the critical behaviour of 2D dilute systems

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Abstract. A continuous expression for the spin-spin correlator of the 2D bond-dilute Ising model is considered. It is shown that its evaluation through straightforward use of RG techniques leads to inconsistent results. Connections with related results in the literature are discussed.

The study of disordered systems attracted a lot of attention some years ago [1]. A criterion—the so-called Harris criterion [2]—has been developed, which allows one to estimate the effects of the introduction of impurities on the critical behaviour of the pure system. However, it cannot give even a qualitative estimate in the case of the 2D Ising model, because the specific heat of this system diverges logarithmically at the critical point. A few years ago, Dotsenko and Dotsenko [3] (hereafter referred to as DD) studied a dilute version of the Ising model and were able to compute the exact leading singularities of thermodynamical quantities. Their results have generated a lot of controversy: the behaviour of the specific heat has been both confirmed [4-7] and questioned [8]; their use of the replica trick has been found to be either invalid [9] or valid [10] by different authors. Recently the behaviour of the spin-spin correlation function was also objected to [7].

In this work we will focus on this last question. We will present a detailed derivation of the result from [3] for the behaviour of the correlation function. Our aim is to show that there is some inconsistency in their calculational method, which manifests itself as a violation of Schwartz's inequality. A brief account of this result, which supports the conclusions of [7], has already been included in a comment [11] to that work; we feel, however, that it is worthwhile to present a complete discussion of the technical questions involved.

The model considered in [3] is the 2D bond-dilute Ising model,

$$\mathscr{H} = -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j \qquad (\sigma_i = \pm 1)$$

where J_{ij} (i, j nearest neighbours) is a random variable with a binary distribution $P(J_{ij}) = (1-c)\delta(J_{ij}-J) + c\delta(J_{ij})$ (c is the concentration of impurities). DD used the replica trick to perform the quenched average over the disorder and, for small c, they mapped the resulting model onto the $N\to 0$ limit of the O(N) Gross-Neveu model:

$$\mathscr{A} = \tfrac{1}{2}\int \mathrm{d}^2x[\bar{\psi}_A(\not{\partial}+m)\psi_A - \tfrac{1}{2}g(\bar{\psi}_A\psi_A)^2] \qquad (A = 1,\ldots,N) \tag{1}$$

with ψ_A a real fermion field, $m \sim (T-T_c)/T_c$ and $g \sim c$. They showed that the disorder-averaged spin-spin correlator at criticality is given by

$$\overline{\langle\sigma_0\sigma_R\rangle} = \left\langle \exp \int_0^R \mathrm{d}x\, \bar{\psi}_1(x)\psi_1(x) \right\rangle \tag{2}$$

where the average on the RHS is taken with respect to the measure defined in (1) with $m=0$. The index 1 to ψ refers to the first replicated fermion species, and the limit $N \to 0$ is implicitly assumed.

By using RG techniques, DD were able to compute the exact leading behaviour of both the specific heat C for $T \sim T_c$, and the correlator in (2) for $R \to \infty$. They found

$$C(\tau) \sim \frac{1}{c} \ln \ln\left(\frac{1}{|\tau|}\right) \qquad \tau = \frac{(T-T_c)}{T_c} \tag{3}$$

and

$$\overline{\langle\sigma_0\sigma_R\rangle} \sim \exp\left(-\frac{\text{constant}}{c}(\ln \ln R)^2\right). \tag{4}$$

The strong modification in the behaviour of the correlator with respect to the pure Ising model ($\langle\sigma_0\sigma_R\rangle \sim R^{-1/4}$) is remarkable since, as DD themselves showed, the introduction of impurities is a marginally attractive perturbation to the pure system. This becomes apparent by studying the flow of the renormalised coupling constant $g(\lambda)$ in the Gross–Neveu model, which for $N \sim 0$ goes like $1/\ln \lambda$ for large scales $\lambda \to \infty$ (see (13) below).

Recently Shankar [7] has reconsidered this problem. Following a simpler route, which involves using bosonisation techniques, he confirmed the result (3) for the specific heat. He was also able to compute $\overline{\langle\sigma_0\sigma_R\rangle^2}$ as given by

$$\overline{\langle\sigma_0\sigma_R\rangle^2} = \left\langle \exp \mathrm{i}\pi \int_0^R \mathrm{d}x\, j_1^0(x) \right\rangle \tag{5}$$

where, due to the squaring, the integration measure corresponds to the U(N) Gross–Neveu model (complex fermions), and $j_1^0 = \bar{\psi}_1\gamma^0\psi_1$ is the fermion charge. Using Schwartz's inequality his result provides a bound for the correlator behaviour:

$$(\overline{\langle\sigma_0\sigma_R\rangle})^2 \leqslant \overline{\langle\sigma_0\sigma_R\rangle^2} \simeq \left(1 + \frac{2}{\pi} g \ln R\right)^{1/4} R^{-1/2} \tag{6}$$

which contradicts the result of DD, (4). Note that in (5) it is assumed that one can replace $\bar{\psi}_1\psi_1$ by j_1^0 in the line integral (compare (2) and (5)), which has been proved to be true in the non-interacting theory [12] (concerning this point see the discussion in [11] and its reply by Shankar). This replacement is needed in order to transform the non-local operator into a two-point correlation function after bosonisation.

The simplicity of Shankar's calculations suggests that there might be a methodological or computational error in the work of DD. In view of this, we were led to restudy their derivation of (4). Those authors employed an early version [13] of the RG approach to obtain their results; we used a slightly different field-theoretical approach following ideas first discussed by Zinn-Justin [14].

We have considered the ensemble average s power ($s = 1, 2$) of the correlation function as given by

$$\overline{\langle\sigma_0\sigma_R\rangle^s} = \left\langle \exp s \int_0^R \mathrm{d}x\, \bar{\psi}_1(x)\psi_1(x) \right\rangle. \tag{7}$$

For $s=1$ this equation coincides with (2); for $s=2$ one has to use complex fermions in computing the average on the RHS, as in (5). Note, however, that we do not change the nature of the operator under the line integral; this will enable us to check the consistency of their calculations and, eventually, to compare the result with the bound given in (6).

By expanding the exponential in (7) in a formal series and taking the logarithm to retain only connected diagrams we get

$$F_s = \ln \overline{\langle \sigma_0 \sigma_R \rangle^s} = \sum_{n=1}^{\infty} \frac{s^n}{n!} \int_0^R \prod_{i=1}^{n} \mathrm{d}x_i \Gamma^{(n)}(x_1 \ldots x_n) \tag{8}$$

where the $\Gamma^{(n)}(x_i)$ are n-point vertex functions of the composite operator $\bar{\psi}_1\psi_1$. For renormalisation purposes we decompose $\bar{\psi}_1\psi_1 = \Omega_1 + \bar{\psi}\cdot\psi$, with the operators $\Omega_1 = \bar{\psi}_1\psi_1 - \bar{\psi}\cdot\psi/N$, and $\bar{\psi}\cdot\psi = \Sigma_A \bar{\psi}_A\psi_A$ transforming independently under irreducible representations of the symmetry group. This leads us to generalise (8) in the form

$$F_s(t_1, t_2) = \sum_{n=1}^{\infty} \frac{s^n}{n!} \sum_{m=0}^{n} \binom{n}{m} \int \prod_{i=1}^{n} \mathrm{d}^2x_i f(x_1) \ldots f(x_n) t_1^m \left(\frac{t_2}{N}\right)^{n-m} \Gamma^{(n,m)}(x_1 \ldots x_n) \tag{9}$$

where $\Gamma^{(n,m)}(x_i)$ contains m insertions of Ω_1 and $(n-m)$ insertions of $\bar{\psi}\cdot\psi$. The function $f(x) = \delta(x_0)\delta(x_1)\theta(R-x_1)$ is introduced to extend the line integral to an integration over the whole 2D space; in the following it can be thought of as a regularising function. Notice that $F_s(t_1=1, t_2=1) = F_s$ and that (9) resembles the expansion of a theory away from the critical point in terms of the critical theory [14].

Going to momentum space and using the RG properties of the cut-off bare functions $\Gamma^{(n,m)}(p_i, \Lambda)$ it is easy to show that $F_s(t_1, t_2)$ satisfies

$$\left\{\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \gamma_\Omega(g) t_1 \frac{\partial}{\partial t_1} - \gamma_{\bar{\psi}\cdot\psi}(g) t_2 \frac{\partial}{\partial t_2}\right\} F_s(t_1, t_2, g, \Lambda) = C_s(g, t_1, t_2) \tag{10}$$

where $C_s(g, t)$ is related to the additive renormalisation constants associated to the $\Gamma^{(2,m)}(p, \Lambda)$ functions (see (15) below and the comment following it). The β and γ functions are defined as usual by

$$\beta(g) = \Lambda \frac{\partial}{\partial\Lambda} g \Big)_{g_R, \mu} \simeq \frac{1}{\pi}(2-sN)g^2 \tag{11a}$$

$$\gamma_\Omega(g) = -\Lambda\frac{\partial}{\partial\Lambda}\ln Z_\Omega \Big)_{g_R, \mu} \simeq -\frac{1}{\pi} g \tag{11b}$$

$$\gamma_{\bar{\psi}\cdot\psi}(g) = -\Lambda\frac{\partial}{\partial\Lambda}\ln Z_{\bar{\psi},\psi} \Big)_{g_R, \mu} \simeq -\frac{1}{\pi}(1-sN)g. \tag{11c}$$

The solution of (10) can be expressed in the standard way in terms of an arbitrary scale parameter λ as

$$F_s(t_1, t_2, g, \Lambda) = F_s\left[t_1(\lambda), t_2(\lambda), g(\lambda), \frac{\Lambda}{\lambda}\right] + \int_1^\lambda \frac{\mathrm{d}x}{x} C_s[g(x), t(x)] \tag{12}$$

with the running couplings

$$t_1(\lambda) = t_1\left[\frac{g(\lambda)}{g}\right]^{1/(2-sN)}$$

$$t_2(\lambda) = t_2\left[\frac{g(\lambda)}{g}\right]^{(1-sN)/(2-sN)}$$

and

$$g(\lambda)=\frac{g}{[1+(1/\pi)(2-sN)g\ln\lambda]}. \tag{13}$$

Notice that, for $N<1$, $t_i(\lambda)$ and $g(\lambda)$ tend to zero at large scales, i.e. we are dealing with an asymptotically free theory in the infrared; this property allows one to obtain the exact leading behaviour of F_s for $R\to\infty$. This can be done simply by expanding the RHS of (12) to second order in $t_i(\lambda)$, which is equivalent to retain only the $n=2$ term in (8) and replacing

$$\Gamma^{(2,m)}(p,g,\Lambda)=\Gamma^{(2,m)}(p,g(\lambda),\Lambda/\lambda)\left[\frac{g(\lambda)}{g}\right]^{\alpha_m(N)}$$
$$-\frac{1}{gs}\frac{(m-1)}{(2-N)}\left[(m-2)N+m\left(1-\frac{1}{N}\right)\right]\frac{\{[g(\lambda)/g]^{\alpha_m(N)-1}-1\}}{[\alpha_m(N)-1]}. \tag{14}$$

To obtain this result we considered the standard RG equations for $\Gamma^{(2,m)}(p,\Lambda)$, which, in addition to the β and γ functions, (12), requires the knowledge of the additive renormalisation constants

$$B^m(g)=\Lambda\frac{\partial}{\partial\Lambda}\Gamma^{(2,m)}(\mu,g,\Lambda)+\mathrm{O}(g)\simeq\frac{(m-1)}{\pi}\left[(m-2)N+m\left(1-\frac{1}{N}\right)\right].$$

In the above approximation, the $N\to 0$ limit of (14) gives

$$\Gamma^{(2,2)}(p,g,\Lambda)+\frac{1}{N^2}\Gamma^{(2,0)}(p,g,\Lambda)$$
$$\simeq\frac{1}{N}\Gamma^{(2,0)}(p,g,\Lambda)\left[\frac{g(\lambda)}{g}\right]\left\{1-s\ln\left[\frac{g(\lambda)}{g}\right]\right\}$$
$$+\frac{1}{gs}\left\{\tfrac{1}{2}s\ln^2\left[\frac{g(\lambda)}{g}\right]-\ln\left[\frac{g(\lambda)}{g}\right]\right\}. \tag{15}$$

Without being particularly explicit in this part of their calculations, DD seem to ignore the second line terms in the above equation, which produce the inhomogeneity in (10). However, they will not change the final result. Going back to coordinate space and choosing the scale parameter $\lambda=|x-y|\Lambda$, with a suitable ultraviolet regularisation for $|x-y|\sim 1/\Lambda$ in the resulting expression we get

$$\ln\overline{\langle\sigma_0\sigma_R\rangle^s}=\frac{s}{(2\pi)^2}\int_0^R\mathrm{d}x\,\mathrm{d}y\frac{\{1+s\ln[1+(2/\pi)g\ln(|x-y|\Lambda)]\}}{[(x-y)^2+\Lambda^{-2}][1+(2/\pi)g\ln(|x-y|\Lambda)]}$$
$$\simeq-\frac{s^2}{4\pi g}(\ln\ln R)^2-\frac{s}{2\pi g}\ln\left(\frac{2e}{\pi}g\right)\ln\ln R. \tag{16}$$

For $s=1$ this is exactly the DD result for the correlator, (4). (In passing we note that, to obtain this behaviour, a linear divergence with R has been dropped out; the same occurs when computing the correlator in the pure Ising model by similar continuum methods [15]. The inhomogeneous terms in (10) contribute to this linear divergence.) For $s=2$ we can see that the Schwartz inequality $(\overline{\langle\sigma_0\sigma_R\rangle})^2\leqslant\overline{\langle\sigma_0\sigma_R\rangle^2}$ is violated, as stated above. This is related to the presence of the factor s multiplying the log-log term in (15), which comes from the renormalisation of t. Notice that the same violation occurs for finite (though small) N, i.e. it is not a problem associated with the $N\to 0$ limit.

It is also interesting to apply the above method to the evaluation of the averaged square of the correlator as given in Shankar's work, (5). One is then led to consider vertex functions of the operator $j_1^0 = \bar{\psi}_1 \gamma^0 \psi_1$. In this case, in the renormalisation of the parameters t, the potentially divergent diagrams vanish due to the presence of the additional γ^0 matrix. In other words, the corresponding anomalous dimension for this operator is zero, which is consistent with the fact that j_1^0 is a conserved current. The expansion equivalent to (8) cannot in principle be cut at $n = 2$; the zero-order terms in $g(\lambda)$ give the pure behaviour $R^{-1/2}$, the contributions coming from $O(g(\lambda))$ terms produce corrections of the form $(\ln R)^\alpha / R^{1/2}$, in agreement with Shankar's result, (6). Note that no inconsistencies appear when considering, for instance, $\langle \sigma_0 \sigma_R \rangle^4$, because no additional factor of two comes from the renormalisation of t unlike in (16).

In conclusion, we have presented a detailed derivation of the results first obtained by DD. We have found that the straightforward use of the RG approach, as employed in [3], gives incorrect results in evaluating the non-local operator in (7). This raises an interesting question concerning the correct use of the RG approach to study the behaviour of such non-local operators. We have also shown that the method gives consistent results in the evaluation of (5), in agreement with independent calculations [7]. The explanation of the above discrepancies constitutes a technical problem which deserves further investigation.

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Note added. After completion of this work we received a preprint of a comment to [7] by A W W Ludwig where the behaviour of the s power of the correlation function $\langle \sigma_0 \sigma_R \rangle^s \sim R^{-s/4} (\ln R)^{s(s-1)/8}$ is reported. This result was found following a different approach, similar to the method developed in [6].

References

[1] Harris A B and Lubensky T C 1974 *Phys. Rev. Lett.* **33** 1540
Khmelnitskii D E 1975 *Sov. Phys.-JETP* **41** 981
[2] Harris A B 1974 *J. Phys. C: Solid State Phys.* **7** 1671
[3] Dotsenko V S and Dotsenko Vl S 1982 *J. Phys. C: Solid State Phys.* **15** 495, L557; 1983 *Adv. Phys.* **32** 129
[4] Wolff W F and Zittartz J 1983 *Z. Phys. B* **52** 117
[5] Jug G 1984 *Phys. Rev. Lett.* **53** 9
[6] Ludwig A W W 1987 *Nucl. Phys.* B **285** [FS19] 97
[7] Shankar R 1987 *Phys. Rev. Lett.* **58** 2466
[8] Ziegler K 1985 *J. Phys. A: Math. Gen.* **18** L801
[9] Ziegler K 1986 *J. Phys. A: Math. Gen.* **19** L943
[10] Murthy G 1987 *Phys. Rev.* B **36** 766
[11] Ceccatto H A and Naon C 1988 *Phys. Rev. Lett* **61** 2389
[12] Itzykson C annd Zuber J B 1977 *Phys. Rev.* D **15** 2875
[13] Larkin A I and Khmelnitskii D E 1969 *Sov. Phys.-JETP* **29** 1123
[14] Zinn-Justin J 1973 *Lectures at Cargese Summer School* unpublished
Brezin E, Le Guillou J C and Zinn-Justin J 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic) p 125
[15] Bander M and Itzykson C 1977 *Phys. Rev.* D **15** 463